

Model Checking and Validity in Propositional and Modal Inclusion Logics

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Abstract. Modal inclusion logic is a formalism that belongs to the family of logics based on team semantics. This article investigates the model checking and validity problems of propositional and modal inclusion logics. We identify complexity bounds for both problems, covering both lax and strict team semantics. Thereby we tie some loose ends related to the programme that ultimately aims to classify the complexities of the basic reasoning problems for modal and propositional dependence, independence, and inclusion logics.

1 Introduction

This paper investigates the complexity of propositional and modal inclusion logic. The focus is on the model checking and validity problems. The complexity of the satisfiability problem of modal inclusion logic was studied by Hella et al. [12], so the current paper directly extends the research effort initiated there.

Modal inclusion logic belongs to the family of modal logics based on *team semantics*. Research on this family of logics has been particularly active in recent years. Team semantics originates from the work of Hodges [16], where it was shown that Hintikka’s IF-logic can be based on a compositional (as opposed to game-theoretic) semantics. Whereas traditional Tarski-style semantics is based on interpreting formulae with respect to variable assignments, team semantics interprets formulae with respect to *sets of assignments*, i.e., teams.

Väänänen introduced a novel variant of IF-logic, called *Dependence Logic* [25]. Dependence Logic was defined using team semantics together with novel operators called *dependence atoms* $=(x_1, \dots, x_k, y)$ which intuitively state that the value of the variable y is functionally determined by the values of x_1, \dots, x_k . Formally, the atom $=(x_1, \dots, x_k, y)$ is true in a team T if for every two assignments $f, g \in T$ it holds that if $f(x_i) = g(x_i)$ for each i , then also $f(y) = g(y)$. Since the introduction of Dependence Logic, a range of variants of that logic based on

different dependency notions have been introduced. The two most widely studied of these variants are *Independence Logic* of Grädel and Väänänen and *Inclusion Logic* of Galliani.

In 2008, Väänänen extended the scope of team semantics to cover modal logic by introducing *Modal Dependence Logic* [26]. Modal dependence logic adds propositional dependence atoms $\text{dep}(p_1, \dots, p_k, q)$ to the syntax of modal logic. A team is now a *set* of possible worlds, and the atom $\text{dep}(p_1, \dots, p_k, q)$ is true in a team T iff each pair of worlds $w, u \in T$ that agree on the truth values of p_1, \dots, p_k , also agree on the truth value of q . In other words, the atom is true iff the truth profile of p_1, \dots, p_k functionally determines the truth value of q .

Also modal variants of Independence Logic and Inclusion Logic (called Modal Independence Logic and Modal Inclusion Logic) have been introduced. For recent related work, see, e.g., [6,14,15,19]. Furthermore, propositional variants of these logics have been introduced and investigated in, e.g., [11,21,?]. The related research programme on modal and propositional logics based on team semantics has concentrated on classifying the complexity and expressivity properties of the involved logics. Due to very active research efforts, the picture concerning the complexities of related logics is understood rather well; see the survey of Durand et al. [5] and the references therein for an overview of the current state of the programme. However, there still are some loose ends. The point of the current article is to tie some of these loose ends. In particular, we will provide complexity bounds for the model checking and validity problems of Propositional Inclusion Logic and Modal Inclusion Logic.

The satisfiability problems of both Propositional and Modal Inclusion Logic were shown EXP-complete in [12], but the complexities of the related validity problems were left open. Below we show that the validity problem is hard for coNEXP for both logics. Concerning model checking, we show that again for both Propositional as well as Modal Inclusion Logic, the problem is complete for P. Furthermore, we additionally investigate the validity and model checking problems of Propositional and Modal Inclusion Logic under *strict semantics*, which is a variant of the standard team semantics (also known as *lax semantics*). Here we show that the model checking problem is NP-complete (for both the modal and propositional cases), whereas for the validity problem of Propositional Inclusion Logic, we establish the same coNP-completeness result that was already known to hold under lax semantics [11].

2 Propositional logics with team semantics

Let D be a finite, possibly empty set of proposition symbols. A function $s: D \rightarrow \{0, 1\}$ is called an *assignment*. A set X of assignments $s: D \rightarrow \{0, 1\}$ is called a *team*. The set D is the *domain* of X . We denote by 2^D the set of *all assignments* $s: D \rightarrow \{0, 1\}$. If $\mathbf{p} = (p_1, \dots, p_n)$ is a tuple of propositions and s is an assignment, we write $s(\mathbf{p})$ for $(s(p_1), \dots, s(p_n))$.

Let Φ be a set of proposition symbols. The syntax of propositional logic $\text{PL}(\Phi)$ is given by the following grammar: $\varphi ::= p \mid \neg p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi)$, where $p \in \Phi$.

We denote by \models_{PL} the ordinary satisfaction relation of propositional logic defined via assignments in the standard way. Next we give team semantics for propositional logic.

Definition 1 (Lax team semantics). *Let Φ be a set of atomic propositions and let X be a team. The satisfaction relation $X \models \varphi$ is defined as follows.*

$$\begin{aligned} X \models p &\Leftrightarrow \forall s \in X : s(p) = 1, \text{ and } X \models \neg p \Leftrightarrow \forall s \in X : s(p) = 0. \\ X \models (\varphi \wedge \psi) &\Leftrightarrow X \models \varphi \text{ and } X \models \psi. \\ X \models (\varphi \vee \psi) &\Leftrightarrow Y \models \varphi \text{ and } Z \models \psi, \text{ for some } Y, Z \text{ such that } Y \cup Z = X. \end{aligned}$$

The lax team semantics is considered the standard semantics for team-based logics. In this paper, we also consider a variant of team semantics called the *strict team semantics*. In strict team semantics, in the above clause for disjunction we require additionally that $Y \cap Z = \emptyset$. When \mathbf{L} denotes a team-based propositional logic, we let \mathbf{L}_s denote the variant of the logic with strict semantics. Thus lax semantics is used unless otherwise specified. The next proposition shows that the team semantics and the ordinary semantics for propositional logic defined via assignments coincide.

Proposition 1 ([25]). *Let φ be a formula of propositional logic and let X be a propositional team. Then $X \models \varphi$ iff $\forall s \in X : s \models_{\text{PL}} \varphi$.*

The syntax of *propositional dependence logic* $\text{PD}(\Phi)$ is obtained by extending the syntax of $\text{PL}(\Phi)$ by the following grammar rule for each $n \in \mathbb{N}$: $\varphi ::= \text{dep}(p_1, \dots, p_n, q)$, where $p_1, \dots, p_n, q \in \Phi$. The semantics for the novel *propositional dependence atoms* is defined as follows: $X \models \text{dep}(p_1, \dots, p_n, q)$ iff for all $s, t \in X : s(p_1) = t(p_1), \dots, s(p_n) = t(p_n)$ implies $s(q) = t(q)$. The next well-known result is proved the same way as the analogous result for first-order dependence logic [25].

Proposition 2 (Downwards closure). *Let φ be a PD-formula and let $Y \subseteq X$ be propositional teams. Then $X \models \varphi$ implies $Y \models \varphi$.*

In this article we study the variant of PD obtained by replacing dependence atoms by so-called *inclusion atoms*: The syntax of *propositional inclusion logic* $\text{Plnc}(\Phi)$ is obtained by extending the syntax of $\text{PL}(\Phi)$ by the grammar rule $\varphi ::= \mathbf{p} \subseteq \mathbf{q}$, where \mathbf{p} and \mathbf{q} are finite tuples of proposition variables with the same length. The semantics for propositional inclusion atoms is defined as follows: $X \models \mathbf{p} \subseteq \mathbf{q}$ iff $\forall s \in X \exists t \in X : s(\mathbf{p}) = t(\mathbf{q})$.

It is easy to check that Plnc is not downward closed logic (cf. Proposition 2). However, analogously to FO-inclusion-logic [8], the same holds for Plnc w.r.t. unions:

Proposition 3 (Closure under unions). *Let $\varphi \in \text{Plnc}$ and let X_i , for $i \in I$, be teams. Suppose that $X_i \models \varphi$ for each $i \in I$. Then $\bigcup_{i \in I} X_i \models \varphi$.*

It is easy to see that due to downward closure, it follows that for propositional logic and propositional dependence logic, the strict and the lax semantics

Satisfiability		Validity		Model checking	
strict	lax	strict	lax	strict	lax
PL	NP [4,18]	coNP [4,18]	NC ¹ [2]		
PD	NP [19]	NEXP [27]	NP [7]		
Plnc	EXP [13]	EXP [12]	coNP [Th. 1]	NP [Th. 3]	P [Th. 2]

Table 1. Complexity of the satisfiability, validity and model checking problems for propositional logics under both systems of semantics. The shown complexity classes refer to completeness results.

coincide (i.e., $PL \equiv PL_s$ and $PD \equiv PD_s$, meaning that same teams satisfy the same formulae in both cases). We will show that for propositional inclusion logic, however, the two different semantics lead to different complexities for the related model checking problems.

3 Complexity of Propositional Inclusion Logic

We now define the model checking, satisfiability, and validity problems in the context of team semantics. Let L be a propositional logic with team semantics. A formula $\varphi \in L$ is *satisfiable*, if there exists a non-empty team X such that $X \models \varphi$. A formula $\varphi \in L$ is *valid*, if $X \models \varphi$ holds for all teams X such that the proposition symbols in φ are in the domain of X . The satisfiability problem $SAT(L)$ and the validity problem $VAL(L)$ are defined in the obvious way: Given a formula $\varphi \in L$, decide whether the formula is satisfiable (valid, respectively). The variant of the model checking problem we study in this article is the following: Given a formula $\varphi \in L$ and a team X , decide whether $X \models \varphi$. See Table 1 for known complexity results for PL, PD, and Plnc, together with partial results of this paper.

It was established in [11] that the validity problem of Plnc is coNP-complete. Here we sketch a proof that the corresponding problem for $Plnc_s$ is also coNP-complete. Our proof is similar to the one in [11]. However the proof of [11] uses the fact that Plnc is union closed, while the same is not true for $Plnc_s$. Lemma 1 is easy to prove by induction. By using Lemma 1, the proof of Theorem 1 is analogous to the corresponding proof of [11] for lax semantics. For details see Appendix A.

Lemma 1. *Let X be a prop. team and $\varphi \in Plnc_s$. If $\{s\} \models \varphi$ for every $s \in X$ then $X \models \varphi$.*

Theorem 1. *The validity problem for $Plnc_s$ is coNP-complete w.r.t. \leq_m^{\log} .*

3.1 Model checking in lax semantics is P-complete

In this section we construct a reduction from the monotone circuit value problem to the model checking problem of Plnc. For a deep introduction to circuits see [28] by Vollmer.

Definition 2. A monotone Boolean circuit with n input gates and one output gate is a 3-tuple $C = (V, E, \alpha)$, where (V, E) is a finite, simple, directed, acyclic graph, and $\alpha: V \rightarrow \{\vee, \wedge, x_1, \dots, x_n\}$ is a function such that the following conditions hold:

1. Every $v \in V$ has in-degree 0 or 2.
2. There exists exactly one $w \in V$ with out-degree 0. We call this node w the output gate of C and denote it by g_{out} .
3. If $v \in V$ is a node with in-degree 0, then $\alpha(v) \in \{x_1, \dots, x_n\}$.
4. If $v \in V$ has in-degree 2, then $\alpha(v) \in \{\vee, \wedge\}$.
5. For each $1 \leq i \leq n$, there exists exactly one $v \in V$ with $\alpha(v) = x_i$.

Let $C = (V, E, \alpha)$ be a monotone Boolean circuit with n input gates and one output gate. Any sequence $b_1, \dots, b_n \in \{0, 1\}$ of bits of length n is called an input to the circuit C . A function $\beta: V \rightarrow \{0, 1\}$ defined such that

$$\beta(v) := \begin{cases} b_i & \text{if } \alpha(v) = x_i \\ \min(\beta(v_1), \beta(v_2)) & \text{if } \alpha(v) = \wedge, \text{ where } (v_1, v), (v_2, v) \in E \text{ and } v_1 \neq v_2 \\ \max(\beta(v_1), \beta(v_2)) & \text{if } \alpha(v) = \vee, \text{ where } (v_1, v), (v_2, v) \in E \text{ and } v_1 \neq v_2 \end{cases}$$

is called the valuation of the circuit C under the input b_1, \dots, b_n . The output of the circuit C is then defined to be $\beta(g_{\text{out}})$.

The *monotone circuit value problem* (MCVP) is the following decision problem: Given a monotone circuit C and an input $b_1, \dots, b_n \in \{0, 1\}$, is the output of the circuit 1?

Proposition 4 ([10]). MCVP is P-complete w.r.t. \leq_m^{\log} reductions.

Lemma 2. MC(Plnc) under lax semantics is P-hard w.r.t. \leq_m^{\log} .

Proof. We will give a LOGSPACE-reduction from MCVP to the model checking problem of Plnc under lax semantics. Since MCVP is P-complete, the claim follows. More precisely, we will show how to construct, for each monotone Boolean circuit C with n input gates and for each input \mathbf{b} for C , a team $X_{C, \mathbf{b}}$ and a Plnc-formula φ_C such that

$$X_{C, \mathbf{b}} \models \varphi_C \text{ iff the output of the circuit } C \text{ with the input } \mathbf{b} \text{ is 1.}$$

We use teams to encode valuations of the circuit. For each gate v_i of a given circuit, we identify an assignment s_i . The crude idea is that if s_i is in the team under consideration, the value of the gate v_i with respect to the given input is 1. The formula φ_C is used to quantify a truth value for each Boolean

gate of the circuit, and then for checking that the truth values of the gates propagate correctly. We next define the construction formally and then discuss the background intuition in more detail.

Let $C = (V, E, \alpha)$ be a monotone Boolean circuit with n input gates and one output gate and let $\mathbf{b} = (b_1 \dots b_n) \in \{0, 1\}^n$ be an input to the circuit C . We define that $V = \{v_0, \dots, v_m\}$ and that v_0 is the output gate of C . Define

$$\tau_C := \{p_0, \dots, p_m, p_\top, p_\perp\} \cup \{p_{k=i \vee j} \mid i < j, \alpha(v_k) = \vee, \text{ and } (v_i, v_k), (v_j, v_k) \in E\}.$$

For each $i \leq m$, we define the assignment $s_i: \tau_C \rightarrow \{0, 1\}$ as follows:

$$s_i(p) := \begin{cases} 1 & \text{if } p = p_i \text{ or } p = p_\top, \\ 1 & \text{if } p = p_{k=i \vee j} \text{ or } p = p_{k=j \vee i} \text{ for some } j, k \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we define $s_\perp(p) = 1$ iff $p = p_\perp$ or $p = p_\top$. We note that the assignment s_\perp will be the only assignment that maps p_\perp to 1. We make use of the fact that for each gate v_i of C , it holds that $s_\perp(p_i) = 0$. We define

$$X_{C, \mathbf{b}} := \{s_i \mid \alpha(v_i) \in \{\wedge, \vee\}\} \cup \{s_i \mid \alpha(v_i) \in \{x_i \mid b_i = 1\}\} \cup \{s_\perp\},$$

that is, $X_{C, \mathbf{b}}$ consists of assignments for each of the Boolean gates, assignments for those input gates that are given 1 as an input, and of the auxiliary assignment s_\perp .

Let X be any nonempty subteam of $X_{C, \mathbf{b}}$ such that $s_\perp \in X$. We have

$$\begin{aligned} X \models p_\top \subseteq p_0 & \text{ iff } s_0 \in X \\ X \models p_i \subseteq p_j & \text{ iff } (s_i \in X \text{ implies } s_j \in X) \\ X \models p_k \subseteq p_{k=i \vee j} & \text{ iff } (i < j, (v_i, v_k), (v_j, v_k) \in E, \alpha(v_k) = \vee \text{ and } s_k \in X \\ & \text{ imply that } s_i \in X \text{ or } s_j \in X) \end{aligned} \tag{1}$$

Recall the intuition that $s_i \in X$ should hold iff the value of the gate v_i is 1. Define

$$\varphi_C := \neg p_\perp \vee (\psi_{\text{output}=1} \wedge \psi_{\text{conjunctions}} \wedge \psi_{\text{disjunctions}}), \tag{2}$$

where

$$\begin{aligned} \psi_{\text{output}=1} &:= p_\top \subseteq p_0, \\ \psi_{\text{conjunctions}} &:= \bigwedge \{p_i \subseteq p_j \mid (v_j, v_i) \in E \text{ and } \alpha(p_i) = \wedge\}, \\ \psi_{\text{disjunctions}} &:= \bigwedge \{p_k \subseteq p_{k=i \vee j} \mid i < j, (v_i, v_k) \in E, (v_j, v_k) \in E, \text{ and } \alpha(v_k) = \vee\}. \end{aligned}$$

It is easy to check that

$$X_{C, \mathbf{b}} \models \varphi_C \text{ iff the output of the circuit } C \text{ with the input } \mathbf{b} \text{ is 1.}$$

The idea of the reduction is the following: The disjunction in (2) is used to guess a team Y for the right disjunct that encodes the valuation β of the circuit

C. The right disjunct is then evaluated with respect to the team Y with the intended meaning that $\beta(v_i) = 1$ whenever $s_i \in Y$. Note that Y is always as required in (1). The formula $\psi_{\text{output}=1}$ is used to state that $\beta(v_0) = 1$, whereas the formulae $\psi_{\text{conjunctions}}$ and $\psi_{\text{disjunctions}}$ are used to propagate the truth value 1 down the circuit. The assignment s_\perp and the proposition p_\perp are used as an auxiliary to make sure that Y is nonempty and to deal with the propagation of the value 0 by the subformulae of the form $p_i \subseteq p_j$.

Now observe that the team $X_{C,b}$ can be easily computed by a logspace Turing machine which scans the input for \wedge -gates, \vee -gates, and true input gates, and then outputs the corresponding team members s_i in a bitwise fashion. The formula φ_C can be computed in logspace as well: (i) the left disjunct does not depend on the input, (ii) for $\psi_{\text{conjunctions}}$ we only need to scan for the \wedge -gates and output the inclusion-formulae for the corresponding edges, (iii) for $\psi_{\text{disjunctions}}$ we need to maintain two binary counters for i and j , and use them for searching for those disjunction gates that satisfy $i < j$. Thus the reduction can be computed in logspace.

In Section 5.1 we will show that the model checking problem for modal inclusion logic with lax semantics is in P (Lemma 5). Since Plnc is essentially a fragment of this logic, by combining Lemmas 2 and 5, we obtain the following theorem.

Theorem 2. *MC(Plnc) under lax semantics is P-complete w.r.t. \leq_m^{\log} .*

3.2 Model checking in lax semantics is NP-complete

In this section we reduce set splitting to model checking problem of Plnc_s .

Definition 3. *The set splitting problem is the following decision problem:*

Input: *A family \mathcal{F} of subsets of a finite set S .*

Problem: *Do there exist subsets S_1 and S_2 of S such that*

1. *S_1 and S_2 are a partition of S (i.e., $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = S$),*
2. *for each $A \in \mathcal{F}$, there exist $a_1, a_2 \in A$ such that $a_1 \in S_1$ and $a_2 \in S_2$?*

Proposition 5 ([9]). *The set splitting problem is NP-complete w.r.t. \leq_m^{\log} .*

Lemma 3. *MC(Plnc) under strict semantics is NP-hard w.r.t. \leq_m^{\log} .*

Proof. We give a reduction from the set splitting problem [9, SP4] to the model checking problem of Plnc under strict semantics.

Let \mathcal{F} be an instance of the set splitting problem. We stipulate that $\mathcal{F} = \{B_1, \dots, B_n\}$ and that $\bigcup \mathcal{F} = \{a_1, \dots, a_k\}$, where $n, k \in \mathbb{N}$. We will introduce fresh proposition symbols p_i and q_j for each point $a_i \in \bigcup \mathcal{F}$ and set $B_j \in \mathcal{F}$. We will then encode the family of sets \mathcal{F} by assignments over these proposition symbols; each assignment s_i will correspond to a unique point a_i . Formally, let

$\tau_{\mathcal{F}}$ denote the set $\{p_1, \dots, p_k, q_1, \dots, q_n, p_{\top}, p_c, p_d\}$ of proposition symbols. For each $i \in \{1, \dots, k, c, d\}$, we define the assignment $s_i: \tau_{\mathcal{F}} \rightarrow \{0, 1\}$ as follows:

$$s_i(p) := \begin{cases} 1 & \text{if } p = p_i \text{ or } p = p_{\top}, \\ 1 & \text{if, for some } j, p = q_j \text{ and } a_i \in B_j, \\ 0 & \text{otherwise.} \end{cases}$$

Define $X_{\mathcal{F}} := \{s_1, \dots, s_k, s_c, s_d\}$, that is, $X_{\mathcal{F}}$ consists of assignments s_i corresponding to each of the points $a_i \in \bigcup \mathcal{F}$ and of two auxiliary assignments s_c and s_d . Note that the only assignment in $X_{\mathcal{F}}$ that maps p_c (p_d , resp.) to 1 is s_c (s_d , resp.) and that every assignment maps p_{\top} to 1. Moreover, note that for $1 \leq i \leq k$ and $1 \leq j \leq n$, $s_i(q_j) = 1$ iff $a_i \in B_j$. Now define

$$\varphi_{\mathcal{F}} := (\neg p_c \wedge \bigwedge_{i \leq n} p_{\top} \subseteq q_i) \vee (\neg p_d \wedge \bigwedge_{i \leq n} p_{\top} \subseteq q_i).$$

We claim that $X_{\mathcal{F}} \models \varphi_{\mathcal{F}}$ iff the output of the set splitting problem with input \mathcal{F} is “yes”.

The proof is straightforward. Note that $X_{\mathcal{F}} \models \varphi_{\mathcal{F}}$ holds iff $X_{\mathcal{F}}$ can be partitioned into two subteams Y_1 and Y_2 such that

$$Y_1 \models \neg p_c \wedge \bigwedge_{i \leq n} p_{\top} \subseteq q_i \text{ and } Y_2 \models \neg p_d \wedge \bigwedge_{i \leq n} p_{\top} \subseteq q_i.$$

Teams Y_1 and Y_2 are both nonempty, since $s_d \in Y_1$ and $s_c \in Y_2$. Also, for a nonempty subteam Y of $X_{\mathcal{F}}$, it holds that $Y \models p_{\top} \subseteq q_j$ iff there exists $s_i \in Y$ such that $s_i(q_j) = 1$, or equivalently, $a_i \in B_j$.

It is now evident that if $X_{\mathcal{F}} \models \varphi_{\mathcal{F}}$ holds then the related subteams Y_1 and Y_2 directly construct a positive answer to the set splitting problem. Likewise, any positive answer to the set splitting problem can be used to directly construct the related subteams Y_1 and Y_2 .

In order to compute the assignments s_i and thus the team $X_{\mathcal{F}}$ on a logspace machine we need to implement two binary counters to count through $1 \leq i \leq k$ for the propositions p_i and $1 \leq j \leq n$ for the propositions q_i . The formula $\varphi_{\mathcal{F}}$ is constructed in logspace by simply outputting it step by step with the help of a binary counter for the interval $1 \leq i \leq n$. Hence the whole reduction can be implemented on a logspace Turing machine.

In Section 5.1 we establish that the model checking problem of modal inclusion logic with strict semantics is in NP (Theorem 5). Since Plnc is essentially a fragment of this logic, together with Lemma 3, we obtain the following theorem.

Theorem 3. *MC(Plnc) under strict semantics is NP-complete w.r.t. \leq_m^{\log} .*

4 Modal logics with team semantics

Let Φ be a set of proposition symbols. The syntax of modal logic $\text{ML}(\Phi)$ is generated by the following grammar: $\varphi ::= p \mid \neg p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Diamond \varphi \mid \Box \varphi$, where $p \in \Phi$. By φ^\perp we denote the formula that is obtained from $\neg \varphi$ by pushing all negation symbols to the atomic level. A (Kripke) Φ -model is a tuple $\mathfrak{M} = (W, R, V)$, where W , called the *domain* of \mathfrak{M} , is a non-empty set, $R \subseteq W \times W$ is a binary relation, and $V : \Phi \rightarrow \mathcal{P}(W)$ is a valuation of the proposition symbols. By \models we denote the *satisfaction relation* of modal logic that is defined via pointed Φ -models in the standard way. Any subset T of the domain of a Kripke model \mathfrak{M} is called a *team* of \mathfrak{M} . Before we define *team semantics* for ML, we introduce some auxiliary notation.

Definition 4. Let $\mathfrak{M} = (W, R, V)$ be a model and T and S teams of \mathfrak{M} . Define that $R[T] := \{w \in W \mid \exists v \in T (vRw)\}$ and $R^{-1}[T] := \{w \in W \mid \exists v \in T (wRv)\}$. For teams T and S of \mathfrak{M} , we write $T[R]S$ if $S \subseteq R[T]$ and $T \subseteq R^{-1}[S]$.

Thus, $T[R]S$ holds if and only if for every $w \in T$, there exists some $v \in S$ such that wRv , and for every $v \in S$, there exists some $w \in T$ such that wRv . We are now ready to define team semantics for ML. We use the symbol “ \models ” for team semantics instead of the symbol “ \Vdash ” which is used for standard Kripke semantics.

Definition 5 (Lax team semantics). Let \mathfrak{M} be a Kripke model and T a team of \mathfrak{M} . The satisfaction relation $\mathfrak{M}, T \models \varphi$ for $\text{ML}(\Phi)$ is defined as follows.

$$\begin{aligned} \mathfrak{M}, T \models p &\Leftrightarrow w \in V(p) \text{ for every } w \in T. \\ \mathfrak{M}, T \models \neg p &\Leftrightarrow w \notin V(p) \text{ for every } w \in T. \\ \mathfrak{M}, T \models (\varphi \wedge \psi) &\Leftrightarrow \mathfrak{M}, T \models \varphi \text{ and } \mathfrak{M}, T \models \psi. \\ \mathfrak{M}, T \models (\varphi \vee \psi) &\Leftrightarrow \mathfrak{M}, T_1 \models \varphi \text{ and } \mathfrak{M}, T_2 \models \psi \text{ for some } T_1 \text{ and } T_2 \\ &\quad \text{such that } T_1 \cup T_2 = T. \\ \mathfrak{M}, T \models \Diamond \varphi &\Leftrightarrow \mathfrak{M}, T' \models \varphi \text{ for some } T' \text{ such that } T[R]T'. \\ \mathfrak{M}, T \models \Box \varphi &\Leftrightarrow \mathfrak{M}, T' \models \varphi, \text{ where } T' = R[T]. \end{aligned}$$

Analogously to the propositional case, we also consider the *strict* variant of team semantics for modal logic. In the *strict* team semantics, we have the following alternative semantic definitions for the disjunction and diamond (where W denotes the domain of \mathfrak{M}).

$$\begin{aligned} \mathfrak{M}, T \models (\varphi \vee \psi) &\Leftrightarrow \mathfrak{M}, T_1 \models \varphi \text{ and } \mathfrak{M}, T_2 \models \psi \text{ for some } T_1 \text{ and } T_2 \\ &\quad \text{such that } T_1 \cup T_2 = T \text{ and } T_1 \cap T_2 = \emptyset. \\ \mathfrak{M}, T \models \Diamond \varphi &\Leftrightarrow \mathfrak{M}, f(T) \models \varphi \text{ for some function } f : T \rightarrow W \\ &\quad \text{such that } \forall w \in T : wRf(w). \end{aligned}$$

When \mathbf{L} is a team-based modal logic, we let \mathbf{L}_s to denote its variant with strict semantics. The formulae of ML have the following flatness property.

Satisfiability		Validity		Model checking	
strict	lax	strict	lax	strict	lax
ML	— PSPACE [17]	— PSPACE [17]	— P	[3,22]	—
MDL	— NEXP [23]	— \in NEXP ^{NP} [27]	— NP	[7]	—
Minc	EXP [13]	EXP [12] coNEXP-h. [C. 2]	coNEXP-h. [L. 7]	NP [T. 5]	P [T. 4]

Table 2. Complexity of satisfiability, validity and model checking for modal logics under both systems of semantics. The given complexity classes refer to completeness results, “-h.” denotes hardness and “ \in ” denotes containment. The complexities for MDL and EMDL as well as Minc and EMinc coincide, see Theorems 4, 5, and 7 and Lemma 9.

Proposition 6 (Flatness). *Let \mathfrak{M} be a Kripke model and T be a team of \mathfrak{M} . Then, for every formula φ of $\text{ML}(\Phi)$: $\mathfrak{M}, T \models \varphi \Leftrightarrow \forall w \in T : \mathfrak{M}, w \Vdash \varphi$.*

The syntax of *modal dependence logic* $\text{MDL}(\Phi)$ and *extended modal dependence logic* $\text{EMDL}(\Phi)$ [6] is obtained by extending the syntax of $\text{ML}(\Phi)$ by the following grammar rule for each $n \in \mathbb{N}$:

$$\varphi ::= \text{dep}(\varphi_1, \dots, \varphi_n, \psi), \text{ where } \varphi_1, \dots, \varphi_n, \psi \in \text{ML}(\Phi).$$

Additionally, for $\text{MDL}(\Phi)$, we require that $\varphi_1, \dots, \varphi_n, \psi$ are proposition symbols in Φ . The semantics for these dependence atoms is defined as follows:

$$\begin{aligned} \mathfrak{M}, T \models \text{dep}(\varphi_1, \dots, \varphi_n, \psi) &\Leftrightarrow \forall w, v \in T : \bigwedge_{1 \leq i \leq n} (\mathfrak{M}, \{w\} \models \varphi_i \Leftrightarrow \mathfrak{M}, \{v\} \models \varphi_i) \\ &\text{implies } (\mathfrak{M}, \{w\} \models \psi \Leftrightarrow \mathfrak{M}, \{v\} \models \psi). \end{aligned}$$

The syntax of *modal inclusion logic* $\text{Minc}(\Phi)$ and *extended modal inclusion logic* $\text{EMinc}(\Phi)$ is obtained by extending the syntax of $\text{ML}(\Phi)$ by the following grammar rule for each $n \in \mathbb{N}$:

$$\varphi ::= \varphi_1, \dots, \varphi_n \subseteq \psi_1, \dots, \psi_n, \text{ where } \varphi_1, \psi_1, \dots, \varphi_n, \psi_n \in \text{ML}(\Phi).$$

Additionally, for $\text{Minc}(\Phi)$, we require that $\varphi_1, \psi_1, \dots, \varphi_n, \psi_n$ are proposition symbols in Φ . The semantics for these inclusion atoms is defined as follows:

$$\begin{aligned} \mathfrak{M}, T \models \varphi_1, \dots, \varphi_n \subseteq \psi_1, \dots, \psi_n \\ \Leftrightarrow \forall w \in T \exists v \in T : \bigwedge_{1 \leq i \leq n} (\mathfrak{M}, \{w\} \models \varphi_i \Leftrightarrow \mathfrak{M}, \{v\} \models \psi_i). \end{aligned}$$

Proposition 7 (Closure properties). *The logics ML, MDL, and EMDL are downward closed. The logics ML, Minc, EMinc are union closed.*

Analogously to the propositional case, it is easy to see that due to downward closure, it follows that for ML, MDL, and EMDL, the strict and the lax semantics coincide. Again, as in the propositional case, this does not hold for Minc or EMinc.

5 Satisfiability, validity and model checking in team semantics

The model checking, satisfiability, and validity problems in the context of team semantics of modal logic are defined analogously to the propositional case. Let $L(\Phi)$ be a modal logic with team semantics. A formula $\varphi \in L(\Phi)$ is *satisfiable*, if there exists a Kripke Φ -model \mathfrak{M} and a non-empty team T of \mathfrak{M} such that $\mathfrak{M}, T \models \varphi$. A formula $\varphi \in L(\Phi)$ is *valid*, if $\mathfrak{M}, T \models \varphi$ holds for every Φ -model \mathfrak{M} and every team T of \mathfrak{M} . The satisfiability problem $\text{SAT}(L)$ and the validity problem $\text{VAL}(L)$ are defined in the obvious way: Given a formula $\varphi \in L$, decide whether the formula is satisfiable (valid, respectively). The variant of the model checking problem that we study in this article is the following: Given a formula $\varphi \in L$, a Kripke model \mathfrak{M} , and a team T of \mathfrak{M} , decide whether $\mathfrak{M}, T \models \varphi$. See Table 2 for known complexity results on ML, MDL, and Minc, together with results of this paper.

5.1 Complexity of model checking

Let \mathfrak{M} be a Kripke model, T be a team of \mathfrak{M} , and φ be a formula of Minc. By $\text{maxsub}(T, \varphi)$, we denote the maximum subteam T' of T such that $\mathfrak{M}, T' \models \varphi$. Since Minc is union closed, such a maximum subteam always exists.

Lemma 4. *If φ is a proposition symbol, its negation, or an inclusion atom, then $\text{maxsub}(T, \varphi)$ can be computed in polynomial time with respect to $|T| + |\varphi|$.*

Proof. If φ is a proposition symbol or its negation, the claim follows from flatness in a straightforward way. Assume then that $T = \{w_1, \dots, w_n\}$ and $\varphi = p_1, \dots, p_k \subseteq q_1, \dots, q_k$. Let $G = (V, E)$ be a directed graph such that $V = T$ and $(u, v) \in E$ iff the value of p_i in u is the same as the value of q_i in v , for each $1 \leq i \leq k$.

The graph G describes the inclusion dependencies between the points in the following sense: if $w \in \text{maxsub}(T, \varphi)$, then there exists some $v \in \text{maxsub}(T, \varphi)$ such that $(w, v) \in E$. Clearly G can be computed in time $\mathcal{O}(n^2k)$. In order to construct $\text{maxsub}(T, \varphi)$, we, round by round, delete all vertices from G with out-degree 0. Formally, we define a sequence G_0, \dots, G_n of graphs recursively. We define that $G_0 := G$ and that G_{j+1} is the graph obtained from G_j by deleting all of those vertices from G_j that have out-degree 0 in G_j . Let i be the smallest integer such that $G_i = (V_i, E_i)$ has no vertices of out-degree 0. Clearly $i \leq n$, and moreover, G_i is computable from G in time $\mathcal{O}(n^3)$. It is easy to check that $V_i = \text{maxsub}(T, \varphi)$.

Lemma 5. *MC(Minc) under lax semantics is in P.*

Proof. We will present a labelling algorithm for model checking $\mathfrak{M}, T \models \varphi$. Let $\text{subOcc}(\varphi)$ denote the set of all *occurrences* of subformulae of φ . Below we denote occurrences as if they were formulae, but we actually refer to some particular occurrence of the formula.

A function $subOcc(\varphi) \rightarrow \mathcal{P}(W)$ is called a labelling function of φ in \mathfrak{M} . We will next give an algorithm for computing a sequence f_0, f_1, f_2, \dots , of such labelling functions.

- Define $f_0(\psi) = W$ for each $\psi \in subOcc(\varphi)$.
- For odd $i \in \mathbb{N}$, define $f_i(\psi)$ bottom up as follows:
 1. For a literal ψ , define $f_i(\psi) := maxsub(f_{i-1}(\psi), \psi)$.
 2. $f_i(\psi \wedge \theta) := f_i(\psi) \cap f_i(\theta)$.
 3. $f_i(\psi \vee \theta) := f_i(\psi) \cup f_i(\theta)$.
 4. $f_i(\Diamond\psi) := \{w \in f_{i-1}(\Diamond\psi) \mid R[w] \cap f_i(\psi) \neq \emptyset\}$.
 5. $f_i(\Box\psi) := \{w \in f_{i-1}(\Box\psi) \mid R[w] \subseteq f_i(\psi)\}$.
- For even $i \in \mathbb{N}$ larger than 0, define $f_i(\psi)$ top to bottom as follows:
 1. Define $f_i(\varphi) := f_{i-1}(\varphi) \cap T$.
 2. If $\psi = \theta \wedge \gamma$, define $f_i(\theta) := f_i(\gamma) := f_i(\theta \wedge \gamma)$.
 3. If $\psi = \theta \vee \gamma$, define $f_i(\theta) := f_{i-1}(\theta) \cap f_i(\theta \vee \gamma)$ and $f_i(\gamma) := f_{i-1}(\gamma) \cap f_i(\theta \vee \gamma)$.
 4. If $\psi = \Diamond\theta$, define $f_i(\theta) := f_{i-1}(\theta) \cap R[f_i(\Diamond\theta)]$.
 5. If $\psi = \Box\theta$, define $f_i(\theta) := f_{i-1}(\theta) \cap R[f_i(\Box\theta)]$.

By a straightforward induction on i , we can prove that $f_{i+1}(\psi) \subseteq f_i(\psi)$ holds for every $\psi \in subOcc(\varphi)$. The only nontrivial induction step is that for $f_{i+1}(\theta)$ and $f_{i+1}(\gamma)$, when $i+1$ is even and $\psi = \theta \wedge \gamma$. To deal with this step, observe that, by the definition of f_{i+1} and f_i , we have $f_{i+1}(\theta) = f_{i+1}(\gamma) = f_{i+1}(\psi)$ and $f_i(\psi) \subseteq f_i(\theta), f_i(\gamma)$, and by the induction hypothesis on ψ , we have $f_{i+1}(\psi) \subseteq f_i(\psi)$.

It follows that there is an integer $j \leq 2 \cdot |W| \cdot |\varphi|$ such that $f_{j+2} = f_{j+1} = f_j$. We denote this fixed point f_j of the sequence f_0, f_1, f_2, \dots by f_∞ . By Lemma 4 the outcome of $maxsub(\cdot, \cdot)$ is computable in polynomial time with respect to its input. Thus clearly f_{i+1} can be computed from f_i in a polynomial number of steps with respect to $|W| + |\varphi|$, whence f_∞ is also computable in polynomial time with respect to $|W| + |\varphi|$.

We will next prove by induction on $\psi \in subOcc(\varphi)$ that $\mathfrak{M}, f_\infty(\psi) \models \psi$. Note first that there is an odd integer i and an even integer j such that $f_\infty = f_i = f_j$.

1. If ψ is a literal, the claim is true since $f_\infty = f_i$ and $f_i(\psi) = maxsub(f_{i-1}(\psi), \psi)$.
2. Assume next that $\psi = \theta \wedge \gamma$, and the claim holds for θ and γ . Since $f_\infty = f_j$, we have $f_\infty(\psi) = f_\infty(\theta) = f_\infty(\gamma)$, whence, by induction hypothesis, $\mathfrak{M}, f_\infty(\psi) \models \theta \wedge \gamma$, as desired.
3. In the case $\psi = \theta \vee \gamma$, we obtain the claim $\mathfrak{M}, f_\infty(\psi) \models \psi$ by using the induction hypothesis, and the observation that $f_\infty(\psi) = f_i(\psi) = f_i(\theta) \cup f_i(\gamma) = f_\infty(\theta) \cup f_\infty(\gamma)$.
4. Assume then that $\psi = \Diamond\theta$. Since $f_\infty = f_i$, we have $f_\infty(\psi) = \{w \in f_{i-1}(\psi) \mid R[w] \cap f_\infty(\theta) \neq \emptyset\}$, whence $f_\infty(\psi) \subseteq R^{-1}[f_\infty(\theta)]$. On the other hand, since $f_\infty = f_j$, we have $f_\infty(\theta) = f_{j-1}(\theta) \cap R[f_\infty(\psi)]$, whence $f_\infty(\theta) \subseteq R[f_\infty(\psi)]$. Thus, $f_\infty(\psi) \models R[f_\infty(\theta)]$, and using the induction hypothesis, we see that $\mathfrak{M}, f_\infty(\psi) \models \psi$.

5. Assume finally that $\psi = \Box\theta$. Since $f_\infty = f_i$, we have $R[f_\infty(\psi)] \subseteq f_\infty(\theta)$. On the other hand, since $f_\infty = f_j$, we have $f_\infty(\theta) \subseteq R[f_\infty(\psi)]$. Thus, $f_\infty(\theta) = R[f_\infty(\psi)]$, whence by the induction hypothesis, $\mathfrak{M}, f_\infty(\psi) \models \psi$.

In particular, if $f_\infty(\varphi) = T$, then $\mathfrak{M}, T \models \varphi$. Thus, to complete the proof of the lemma, it suffices to prove that the converse implication is true, as well. To prove this, assume that $\mathfrak{M}, T \models \varphi$. Then for each $\psi \in \text{subOcc}(\varphi)$, there is a team T_ψ such that

1. $T_\varphi = T$.
2. If $\psi = \theta \wedge \gamma$, then $T_\psi = T_\theta = T_\gamma$.
3. If $\psi = \theta \vee \gamma$, then $T_\psi = T_\theta \cup T_\gamma$.
4. If $\psi = \Diamond\theta$, then $T_\psi[R]T_\theta$.
5. If $\psi = \Box\theta$, then $T_\theta = R[T_\psi]$.
6. If ψ is a literal, then $\mathfrak{M}, T_\psi \models \psi$.

We prove by induction on i that $T_\psi \subseteq f_i(\psi)$ for all $\psi \in \text{subOcc}(\varphi)$. For $i = 0$, this is obvious, since $f_0(\psi) = W$ for all ψ . Assume next that $i + 1$ is odd and the claim is true for i . We prove the claim $T_\psi \subseteq f_i(\psi)$ by induction on ψ .

1. If ψ is a literal, then $f_{i+1}(\psi) = \text{maxsub}(f_i(\psi), \psi)$. Since $\mathfrak{M}, T_\psi \models \psi$, and by induction hypothesis, $T_\psi \subseteq f_i(\psi)$, the claim $T_\psi \subseteq f_{i+1}(\psi)$ is true.
2. Assume that $\psi = \theta \wedge \gamma$. By induction hypothesis on θ and γ , we have $T_\psi = T_\theta \subseteq f_{i+1}(\theta)$ and $T_\psi = T_\gamma \subseteq f_{i+1}(\gamma)$. Thus, we get $T_\psi \subseteq f_{i+1}(\theta) \cap f_{i+1}(\gamma) = f_{i+1}(\psi)$.
3. The case $\psi = \theta \vee \gamma$ is similar to the previous one; we omit the details.
4. If $\psi = \Diamond\theta$, then $f_{i+1}(\psi) = \{w \in f_i(\psi) \mid R[w] \cap f_{i+1}(\theta) \neq \emptyset\}$. By the two induction hypotheses on i and θ , we have $\{w \in T_\psi \mid R[w] \cap T_\theta \neq \emptyset\} \subseteq f_{i+1}(\psi)$. The claim follows from this, since the condition $R[w] \cap T_\theta \neq \emptyset$ holds for all $w \in T_\psi$.
5. The case $\psi = \Box\theta$ is again similar to the previous one, so we omit the details.

Assume then that $i + 1$ is even and the claim is true for i . This time we prove the claim $T_\psi \subseteq f_i(\psi)$ by top to bottom induction on ψ .

1. By assumption, $T_\varphi = T$, whence by induction hypothesis, $T_\varphi \subseteq f_i(\varphi) \cap T = f_{i+1}(\varphi)$.
2. Assume that $\psi = \theta \wedge \gamma$. By induction hypothesis on ψ , we have $T_\psi \subseteq f_{i+1}(\psi)$. Since $T_\psi = T_\theta = T_\gamma$ and $f_{i+1}(\psi) = f_{i+1}(\theta) = f_{i+1}(\gamma)$, this implies that $T_\theta \subseteq f_{i+1}(\theta)$ and $T_\gamma \subseteq f_{i+1}(\gamma)$.
3. Assume that $\psi = \theta \vee \gamma$. Using the fact that $T_\theta \subseteq T_\psi$, and the two induction hypotheses on i and ψ , we see that $T_\theta \subseteq f_i(\theta) \cap T_\psi \subseteq f_i(\theta) \cap f_{i+1}(\psi) = f_{i+1}(\theta)$. Similarly, we see that $T_\gamma \subseteq f_{i+1}(\gamma)$.
4. Assume that $\psi = \Diamond\theta$. By the induction hypothesis on i , we have $T_\theta \subseteq f_i(\theta)$, and by the induction hypothesis on ψ , we have $T_\theta \subseteq R[T_\psi] \subseteq R[f_{i+1}(\psi)]$. Thus, we see that $T_\theta \subseteq f_i(\theta) \cap R[f_{i+1}(\psi)] = f_{i+1}(\theta)$.
5. The case $\psi = \Box\theta$ is similar to the previous one; we omit the details.

It follows now that $T = T_\varphi \subseteq f_\infty(\varphi)$. Since $f_\infty(\varphi) \subseteq f_2(\varphi) \subseteq T$, we conclude that $f_\infty(\varphi) = T$. This completes the proof of the implication $\mathfrak{M}, T \models \varphi \Rightarrow f_\infty(\varphi) = T$.

Lemma 6. *MC(EMinc) under lax semantics is in P.*

Proof. The result follows by a polynomial time reduction to the model checking problem of Minc. Simply put, we introduce fresh proposition symbols p_i for each ML-formula ϕ_i that occurs as a parameter of some inclusion atom. We make sure that the extensions of p_i and ϕ_i coincide and then substitute ϕ_i s by p_i s. For details see Appendix B.

By combining Lemmas 2, 5, and 6, we obtain the following theorem.

Theorem 4. *MC(Minc) and MC(EMinc) under lax semantics are P-complete w.r.t. \leq_m^{\log} .*

MC(Minc) and MC(EMinc) under strict semantics can be shown to be in NP by a brute force algorithm, see Appendix B. Thus by Lemma 3, we obtain the following theorem.

Theorem 5. *MC(Minc) and MC(EMinc) under strict semantics are NP-complete w.r.t. \leq_m^{\log} .*

5.2 Dependency quantifier Boolean formulae

Deciding whether a given quantified Boolean formula (qBf) is valid is a canonical PSPACE-complete problem. *Dependency quantifier Boolean formulae* introduced by Peterson et al. [20] are variants of qBfs for which the corresponding decision problem is NEXP-complete. In this section, we define the related coNEXP-complete complement problem. For the definitions related to dependency quantifier Boolean formulae, we follow Virtema [27].

QBfs extend propositional logic by allowing a prenex quantification of proposition symbols. Formally, the set of *qBfs* is built from formulae of propositional logic by the following grammar: $\varphi ::= \exists p \varphi \mid \forall p \varphi \mid \theta$, where p is a propositional variable (i.e., a proposition symbol) and θ is formula of propositional logic. The semantics for qBfs is defined via assignments $s: \text{PROP} \rightarrow \{0, 1\}$ in the obvious way. When C is a set of propositional variables, we denote by \mathbf{c} the canonically ordered tuple of the variables in the set C . When p is a propositional variable and $b \in \{0, 1\}$ is a truth value, we denote by $s(p \mapsto b)$ the modified assignment defined as follows:

$$s(p \mapsto b)(q) := \begin{cases} b & \text{if } q = p, \\ s(q) & \text{otherwise.} \end{cases}$$

A formula that does not have any free variables is called *closed*. We denote by QBF the set of exactly all closed quantified Boolean formulae.

Proposition 8 ([24]). *The validity problem of QBF is PSPACE-complete w.r.t. \leq_m^{\log} .*

A *simple qBf* is a closed qBf of the type $\varphi := \forall p_1 \cdots \forall p_n \exists q_1 \cdots \exists q_k \theta$, where θ is a propositional formula and the propositional variables p_i, q_j are all distinct. Any tuple (C_1, \dots, C_k) such that $C_1, \dots, C_k \subseteq \{p_1, \dots, p_n\}$ is called a *constraint* for φ . Intuitively, a constraint $C_j = \{p_1, p_3\}$ can be seen as a dependence atom $\text{dep}(p_1, p_3, q_j)$. Note that, without loss of generality, one can assume that for each constraint (C_1, \dots, C_k) it holds that $|C_i| = |C_j|$ for $1 \leq i \neq j \leq k$ (by introducing fresh dummy variables). The constraints that have the previously mentioned property are called *normalised*.

Definition 6. A simple qBf $\forall p_1 \cdots \forall p_n \exists q_1 \cdots \exists q_k \theta$ is valid under a constraint (C_1, \dots, C_k) , if there exist functions f_1, \dots, f_k with $f_i: \{0, 1\}^{|C_i|} \rightarrow \{0, 1\}$ such that for each assignment $s: \{p_1, \dots, p_n\} \rightarrow \{0, 1\}$, $s(q_1 \mapsto f_1(s(c_1)), \dots, q_k \mapsto f_k(s(c_k))) \models \theta$.

A *dependency quantifier Boolean formula* is a pair (φ, \mathbf{C}) , where φ is a simple quantified Boolean formula and \mathbf{C} is a normalised constraint for φ . We say that (φ, \mathbf{C}) is *valid*, if φ is valid under the constraint \mathbf{C} . Let DQBF denote the set of all dependency quantifier Boolean formulae.

Proposition 9 ([20, 5.2.2]). *The validity problem of DQBF is NEXP-complete w.r.t. \leq_m^{\log} .*

Definition 7. Given a simple qBf $\forall p_1 \cdots \forall p_n \exists q_1 \cdots \exists q_k \theta$, we say it is non-valid under a constraint (C_1, \dots, C_k) , if for all functions f_1, \dots, f_k with $f_i: \{0, 1\}^{|C_i|} \rightarrow \{0, 1\}$, there exists an assignment $s: \{p_1, \dots, p_n\} \rightarrow \{0, 1\}$ such that $s(q_1 \mapsto f_1(s(c_1)), \dots, q_k \mapsto f_k(s(c_k))) \not\models \theta$.

It is straightforward to see that non-validity problem of DQBF is the complement problem of the validity problem of DQBF. Thus the following corollary follows.

Corollary 1. *The non-validity problem of DQBF is coNEXP-complete w.r.t. \leq_m^{\log} .*

5.3 Complexity of the validity problem is coNEXP-hard

In Section 5.3 we give a reduction from the non-validity problem of DQBF to the validity problem of Minc.

Lemma 7. *VAL(Minc) under lax semantics is coNEXP-hard w.r.t. \leq_m^{\log} .*

Proof. We provide a \leq_m^{\log} -reduction from the non-validity problem of DQBF (which is by Corollary 1 complete for coNEXP w.r.t. \leq_m^{\log}) to the validity problem of Minc.

Recall Definition 7. In our reduction we will encode all the possible modified assignments of Definition 7 by points in a Kripke model. First we enforce binary (assignment) trees in our structures. Each leaf of the binary tree will correspond to some possible witness for the assignment s of Definition 7. The binary trees are forced in the standard way by modal formulae: The formula $\text{branch}(p_i) := \Diamond p_i \wedge$

$\Diamond \neg p_i$ forces that there are ≥ 2 successor states which disagree on a proposition p_i . The formula $\text{store}(p_i) := (p_i \rightarrow \Box p_i) \wedge (\neg p_i \rightarrow \Box \neg p_i)$ is used to propagate chosen values for p_i to successors in the tree. Now define

$$\text{tree}(p, n) := \text{branch}(p_1) \wedge \bigwedge_{i=1}^{n-1} \Box^i \left(\text{branch}(p_{i+1}) \wedge \bigwedge_{j=1}^i \text{store}(p_j) \right),$$

where $\Box^i \varphi := \overbrace{\Box \cdots \Box}^{i \text{ many}} \varphi$ is the i -times concatenation of \Box . The formula $\text{tree}(p, n)$ forces a complete binary assignment tree of depth n for proposition symbols p_1, \dots, p_n . Notice that $\text{tree}(p, n)$ is an ML-formula and thus flat (see Proposition 6).

Recall again Definition 7 and consider the simple qBf $\forall p_1 \cdots \forall p_n \exists q_1 \cdots \exists q_k \theta$ with constraint (C_1, \dots, C_k) . The construction above will be used to deal with the proposition symbols p_i . Values for the functions f_j and thus for the values of proposition symbols q_j arise from particular models; essentially since we are considering validity all possible values will be considered. We will next define a formula that will deal with those models in which, for some j , the values for q_j do not respect the related constraint C_j , i.e., that does not give rise to a function f_j in Definition 7. These unwanted models have to be “filtered” out by the formula by satisfaction. This violation is expressed via φ_{cons} defined as follows:

$$\varphi_{\text{cons}} := \bigvee_{\substack{1 \leq j \leq k, \\ C_j = \{p_{i_1}, \dots, p_{i_\ell}\}}} (t_1 \cdots t_\ell t_\perp \subseteq p_{i_1} \cdots p_{i_\ell} q_j) \wedge (t_1 \cdots t_\ell t_\top \subseteq p_{i_1} \cdots p_{i_\ell} q_j). \quad (3)$$

For the time being, suppose that the values for the proposition symbols t_i have been existentially quantified, and that t_\top and t_\perp correspond to the constant values 1 and 0, respectively (we will later show how this is technically done). Now the formula φ_{cons} essentially states that there exists a q_j that does not respect the constraint C_j . Now define

$$\begin{aligned} \varphi_{\text{struc}} := & \text{tree}(p, n) \wedge \Box^n (\text{tree}(t, \ell)) \wedge \Box^{n+\ell} ((p_\theta \leftrightarrow \theta) \wedge p_\top \wedge \neg p_\perp) \\ & \wedge \Box^n \left(\bigwedge_{1 \leq i \leq \ell} \Box^i \left(\bigwedge_{1 \leq j \leq n} \text{store}(p_j) \right) \right). \end{aligned}$$

The formula φ_{struc} enforces the full binary assignment tree w.r.t. the p_i s, enforces in their leaves trees of depth ℓ for variables t_i (necessary for formula φ_{cons}), identifies the truth of θ by a proposition p_θ at the depth $n + \ell$ as well as 1 by t_\top and 0 by t_\perp , and then stores the values of the p_i s consistently in their subtrees of relevant depth. Finally define

$$\varphi_{\text{non-val}} := \varphi_{\text{struc}}^\perp \vee \left(\varphi_{\text{struc}} \wedge \Box^n (\Diamond^\ell (\varphi_{\text{cons}} \vee p_\perp \subseteq p_\theta)) \right). \quad (4)$$

By $\varphi_{\text{struc}}^\perp$, we denote the negation normal form of the ML-formula $\neg \varphi_{\text{struc}}$. An important observation is that since φ_{struc} is an ML-formula, it is flat. Note

that whereas the values for the proposition symbols p_i are fixed in the level n of the tree (and are then propagated to successors), the values for the proposition symbols q_j are decided only at the final depth, that is, at the depth $n + \ell$. Now the formula $\varphi_{\text{non-val}}$ is valid if and only if

$$\mathfrak{M}, T \models \Box^n (\Diamond^\ell (\varphi_{\text{cons}} \vee p_\perp \subseteq p_\theta)) \quad (5)$$

holds for every team pointed Kripke model \mathfrak{M}, T that satisfies the structural properties forced by φ_{struc} . Let us now return to the formula (3). There we assumed that the proposition symbols t_i had been quantified and that the symbols p_\top and p_\perp correspond to the logical constants. The latter part we already dealt with in the formula φ_{struc} . Recall that φ_{struc} forces full binary assignment trees for the t_i s that start from depth n . The quantification of the t_i s is done by selecting the corresponding successors by the diamonds \Diamond^ℓ in the formula (4). If \mathfrak{M}, T is such that, for some j , q_j does not respect the constraint C_j , we use \Diamond^ℓ to guess a witness for the violation. It is then easy to check that the whole team obtained by evaluating the diamond prefix satisfies the formula φ_{cons} . On the other hand, if \mathfrak{M}, T is such that for each j the value of q_j respects the constraint C_j , then the subformula $p_\perp \subseteq p_\theta$ forces that there exists a point w in the team obtained from T by evaluating the modalities in (5) such that $\mathfrak{M}, \{w\} \not\models p_\theta$. In our reduction this means that w gives rise a propositional assignment that falsifies θ as required in Definition 7.

It is now straightforward to show that a simple qBf $\forall p_1 \dots \forall p_n \exists q_1 \dots \exists q_k \theta$ is *non-valid under a constraint* (C_1, \dots, C_k) iff the Minc-formula $\varphi_{\text{non-val}}$ obtained as described above is valid. We skip further details.

In order to compute $\varphi_{\text{non-val}}$ two binary counters bounded above by $n + k + \ell$ need to be maintained. Note that $\log(n + k + \ell)$ is logarithm with respect to the input length. Hence the reduction is computable in logspace wherefore the lemma applies.

Clearly the previous proof works also for strict semantics. Thus we obtain the following.

Corollary 2. *VAL(Minc) under strict semantics is coNEXP-hard w.r.t. \leq_m^{\log} .*

6 Conclusion

In this paper we investigated the computational complexity of model checking and validity for propositional and modal inclusion logic in order to complete the complexity landscape of these problems in the mentioned logics. In particular we emphasise on the subtle influence of which semantics are considered: strict or lax. Both logics' complexity behaves similar with respect to model checking, i.e., under strict semantics it is NP- and under lax semantics P-complete. The validity problem is shown to be coNP-complete for the propositional strict semantics case. For the modal case we achieve a coNEXP lower bound under lax as well as strict semantics. The upper bound is left open for further research. It is however

easy to establish that, if closed under polynomial reductions, the complexities of $\text{VAL}(\text{Minc})$ and $\text{VAL}(\text{EMinc})$, and $\text{VAL}(\text{Minc}_s)$ and $\text{VAL}(\text{EMinc}_s)$ coincide, respectively, see Appendix B.

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A The validity problem of the strict variant of PInc is coNP-complete

It was established in [11] that the validity problem of PInc is coNP-complete. Here we show that the corresponding problem for PInc_s is also coNP-complete.

Our proof is similar to the one in [11]. However the proof of [11] uses the fact that PInc is union closed, while the same is not true for PInc_s.

Lemma 8. *Let X be a propositional team and $\varphi \in \text{PInc}_s$. If $\{s\} \models \varphi$ for every $s \in X$ then $X \models \varphi$.*

Proof. The proof is by a simple induction on the structure of the formula. The cases for atomic formulae and conjunction are trivial. The case for disjunction is easy: Assume that $\{s\} \models \varphi \vee \psi$ for every $s \in X$. Thus for every $s \in X$ either $\{s\} \models \varphi$ or $\{s\} \models \psi$. Therefore there exists Y and Z such that $Y \cup Z = X$, $Y \cap Z = \emptyset$, $\forall s \in Y : \{s\} \models \varphi$, and $\forall s \in Z : \{s\} \models \psi$. By the induction hypothesis $Y \models \varphi$ and $Z \models \psi$. Thus $X \models \varphi \vee \psi$.

Theorem 6. *The validity problem for PInc_s is coNP-complete.*

Proof. Since PL can be seen as a sublogic of PInc_s, the coNP-harness follows from the fact that the validity problem of PL is coNP-complete. Therefore, it suffices to show $\text{VAL}(\text{PInc}_s) \in \text{coNP}$. It is easy to check that, by Lemma 8, a formula $\varphi \in \text{PInc}_s$ is valid iff it is satisfied by all singleton teams $\{s\}$. Note also that, over a singleton team $\{s\}$, an inclusion atom $(p_1, \dots, p_n) \subseteq (q_1, \dots, q_n)$ is equivalent to the PL-formula

$$\bigwedge_{1 \leq i \leq n} (p_i \wedge q_i) \vee (\neg p_i \wedge \neg q_i).$$

Denote by φ^* the PL-formula obtained by replacing all inclusion atoms in φ by their PL-translations. By the above, φ is valid iff φ^* is valid. Since $\text{VAL}(\text{PL})$ is in coNP the claim follows.

B Complexity of Modal Inclusion Logic

Lemma 9. *MC(EMinc) under lax semantics is in P.*

Proof. The result follows by a polynomial time reduction to the model checking problem of Minc: Let $(W, R, V), T$ be a team pointed Kripke model and φ be a formula of EMinc. Let $\varphi_1, \dots, \varphi_n$ be exactly those subformulae of φ that occur as a parameter of some inclusion atom in φ and let p_1, \dots, p_n be distinct fresh proposition symbols. Let V' be a valuation defined as follows

$$V'(p) := \begin{cases} \{w \in W \mid (W, R, V), w \Vdash \varphi_i\} & \text{if } p = p_i, \\ V(p) & \text{otherwise.} \end{cases}$$

Let φ^* denote the formula obtained from φ by simultaneously substituting each φ_i by p_i . It is easy to check that $(W, R, V), T \models \varphi$ if and only if $(W, R, V'), T \models \varphi^*$. Moreover φ^* can be clearly computed from φ in polynomial time. Likewise V' can be computed in polynomial time; since each φ_i is a modal formula the truth set of that formula in (W, R, V) can be computed in polynomial time by the standard labelling algorithm used in modal logic (see e.g., [1]), and the numbers of such computations is bounded above by the size of φ . Thus the result follows from Lemma 5.

Lemma 10. *MC(Minc) and MC(EMinc) under strict semantics are in NP.*

Proof. The obvious brute force algorithm for model checking for EMinc works in NP: For disjunctions and diamonds, we use nondeterminism to guess the correct partitions or successor teams, respectively. Conjunctions are dealt sequentially and for boxes the unique successor team can be computed by brute force in quadratic time. Checking whether a team satisfies an inclusion atom or a (negated) proposition symbol can be computed by brute force in polynomial time (this also follows directly from Lemma 4).

Theorem 7. *Let \mathbf{C} be a complexity class that is closed under polynomial time reductions. Then VAL(Minc) under lax (strict) semantics is complete for \mathbf{C} if and only if VAL(EMinc) under lax (strict) semantics is complete for \mathbf{C} .*

Proof. Let φ be a formula of EMinc. Let $\varphi_1, \dots, \varphi_k$ be exactly those subformulae of φ that occur as a parameter of some inclusion atom in φ and let p_1, \dots, p_n be distinct fresh proposition symbols. Define

$$\varphi_{subst} := \left(\bigwedge_{0 \leq i \leq k} \Box^i \bigwedge_{1 \leq j \leq n} (p_j \leftrightarrow \psi_j) \right), \quad \varphi^* := \varphi_{subst}^\perp \vee (\varphi_{subst} \wedge \varphi^+),$$

where φ_{subst}^\perp denotes the negation normal form of $\neg \varphi_{subst}$ and φ^+ is the formula obtained from φ by simultaneously substituting each φ_i by p_i . It is easy to check that φ is valid if and only if the Minc formula φ^* is. Clearly φ^* is computable from φ in polynomial time.